ON THE REDUCTION OF A SYSTEM OF LINEAR DIFFERENTIAL FORMS OF ANY ORDER.

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In the classical theory of the extremum of a definite integral of the form $\int F(x, y, x', y')dt$, the discussion of the second variation and of Jacobi's differential equation is based on the reduction of the second variation by means of Weierstrass's transformation to a form analogous to that of the second variation of the integral $\int f(x, y, y')dx$.* It appears to the writer that the essential character of Weierstrass's transformation is the reduction of linear differential forms of the second order in three variables to similar forms in two variables. This character becomes more apparent when one attempts to parallel Weierstrass's work for the case of an integrand containing n unknown functions of t and their first derivatives.

The purpose of the present note is to consider, from a general point of view, the problem of reducing a system of n linearly independent differential forms of pth order, containing $n + 1$ unknown functions of t, to a similar system containing n unknown functions. The application to the calculus of variations can then be made by putting $p = 2$. So far as I have been able to determine, this problem has not previously been discussed in the literature.

Suppose we have n differential forms:

$$(1) \qquad \psi_i \equiv \sum_{j=1}^{n+1} \sum_{l=0}^{p} a_{ilj} x_j^{(l)} \qquad (i = 1 \cdots n),$$

where a_{ilj} are functions of t of class $C^{(l)}$† on an interval (t_1t_2) and such that the matrix $\| a_{ipj} \|$ is of rank n.‡

We shall then seek to determine α_{kj} and A_{ilk} in such a way that we can write

$$(2) \qquad \psi_i \equiv \sum_{k=1}^{n} \sum_{l=0}^{p} A_{ilk} y_k^{(l)},$$

where

$$(3) \qquad y_k \equiv \sum_{j=1}^{n+1} \alpha_{kj} x_j.$$

* Compare, e. g., Bolza, Vorlesungen über Variationsrechnung, §§ 28 and 29.

† A function is said to be of class $C^{(l)}$ if the function is continuous and has continuous derivatives of the 1st, 2d, . . ., lth order.

‡ For the definition of rank, see Bôcher, Introduction to Higher Algebra, p. 22.

If we suppose $|A_{ipk}| \neq 0$, and put

$$(4) \qquad z_i = \sum_{k=1}^{n} A_{ipk} y_k,$$

we can solve the latter equations for y_k in terms of z_i; substituting the results in (2) would reduce these expressions to the form

$$(5) \qquad \psi_i \equiv z_i^{(p)} + \sum_{k=1}^{n} \sum_{h=0}^{p-1} B_{ihk} z_k^{(h)};$$

and, substitution of (4) in (5) would reduce these back to (2).

Hence there is no loss in generality if we try to determine β_{kj} and B_{ihk} in such a way as to reduce (1) to the form (5) directly, where

$$(6) \qquad z_k \equiv \sum_{j=1}^{n+1} \beta_{kj} x_j.$$

Substituting (6) in (5) and comparing the coefficients of the terms in $x_j^{(p)}$, with the corresponding coefficients in (1), we see at once that

$$\beta_{kj} \equiv a_{kpj}$$

for all values of k and j. Thence we have

$$(7) \qquad z_i = \sum_{j=1}^{n+1} a_{ipj} x_j.$$

Rather than compare coefficients of terms of lower order, we proceed now as follows: Since the matrix $\| a_{ipj} \|$ was supposed to be of rank n, equations (7) can be solved for x_j in terms of z_i; we find:

$$x_j = \sum_{k=1}^{n} C_{ji} z_i + \Delta_j w,$$

where Δ_j is the n-rowed determinant obtained from $\| a_{ipj} \|$ by striking out the jth column and multiplying the result by $(-1)^{j-1}$, w being an arbitrary function of t.

If we substitute these results in (1), these forms will reduce to forms (5), *if, and only if, the terms in w vanish identically.*

This leads at once to the following necessary and sufficient conditions for the possibility of our reduction:

$$(8) \qquad \sum_{j=0}^{n+1} \sum_{l=0}^{p} a_{ilj} (\Delta_j w)^{(l)} \underset{w}{\equiv} 0, \qquad (i = 1, \cdots, n).$$

In order to express these conditions in terms of the coefficients of (1) alone, we make use of the following formula:

$$uv^{(l)} \equiv \sum_{m=0}^{l} (-1)^m C_{l,\,m} (u^{(m)} v)^{(l-m)}, \tag{9}$$

where

$$C_{l,\,m} \equiv \frac{l!}{m!\,(l-m)!},$$

which we can prove by a complete induction.

We know $uv^{(l)} \equiv (uv^{(l-1)})' - u'v^{(l-1)}$.

Assuming the formula to hold for $l - 1$, we have:

$$uv^{(l-1)} \equiv \sum_{m=0}^{l-1} (-1)^m C_{l-1,\,m} (u^{(m)} v)^{(l-m-1)};$$

$$u'v^{(l-1)} \equiv \sum_{m=0}^{l-1} (-1)^m C_{l-1,\,m} (u^{(m+1)} v)^{(l-m-1)}$$

$$\equiv \sum_{m=1}^{l} (-1)^{m-1} C_{l-1,\,m-1} (u^{(m)} v)^{(l-m)}.$$

Hence

$$uv^{(l)} \equiv \sum_{m=0}^{l-1} (-1)^m C_{l-1,\,m} (u^{(m)} v)^{(l-m)} + \sum_{m=1}^{l} (-1)^m C_{l-1,m-1} (u^{(m)} v)^{(l-m)}$$

$$\equiv \sum_{m=0}^{l} (-1)^m [C_{l-1,\,m} + C_{l-1,\,m-1}] (u^{(m)} v)^{(l-m)}$$

$$\equiv \sum_{m=0}^{l} (-1)^m C_{l,\,m} (u^{(m)} v)^{(l-m)},$$

which proves formula (9) for l. Since we can verify by immediate substitution that the formula holds for $l = 2, 3$, its validity has been established.

By means of (9), conditions (8) reduce to:

$$\sum_{j=1}^{n+1} \sum_{l=0}^{p} \sum_{m=0}^{l} (-1)^m C_{l,\,m} [a_{ilj}{}^{(m)} \Delta_j w]^{(l-m)} \equiv 0, \qquad (i = 1 \cdots n).$$

Putting $l - m = r$, these become

$$\sum_{j=1}^{n+1} \sum_{r=0}^{p} \sum_{l=r}^{p} (-1)^{l-r} C_{l,\,l-r} [a_{ilj}{}^{(l-r)} \Delta_j w]^{(r)} \equiv 0, \qquad (i = 1 \cdots n).$$

Again, since these conditions have to hold true for any function w, we obtain the following $pn + n$ conditions,

$$\sum_{j=1}^{n+1} \sum_{l=r}^{p} (-1)^{l-r} C_{l,\,l-r} a_{ilj}{}^{(l-r)} \Delta_j \equiv 0 \qquad (r = 0, \cdots, p;\ i = 1, \cdots, n). \tag{10}$$

For $r = p$, these reduce to

$$\sum_{j=1}^{n+1} a_{ipj} \Delta_j \equiv 0 \qquad (i = 1, \cdots, n),$$

which are satisfied in virtue of the definition of Δ_j. The remaining pn conditions can be expressed by equating to zero certain $(n+1)$-rowed determinants. The first n rows of these are always formed by the matrix $\| a_{ipj} \|$; the last row is formed by the elements

$$\sum_{l=r}^{p} (-1)^{l-r} C_{l,\, l-r} a_{ilj}{}^{(l-r)} \qquad (j = 1, \cdots, n+1).$$

For the particular case of 2 forms of the second order involving 3 unknown functions of t, the conditions take the form

$$\begin{vmatrix} a_{131} & a_{132} & a_{133} \\ a_{231} & a_{232} & a_{233} \\ 2a_{i31}' - a_{i21} & 2a_{i32}' - a_{i22} & 2a_{i33}' - a_{i23} \end{vmatrix} = 0 \qquad (i = 1,\ 2),$$

$$\begin{vmatrix} a_{131} & a_{132} & a_{133} \\ a_{231} & a_{232} & a_{233} \\ a_{i31}'' - a_{i21}' + a_{i11} & a_{i32}'' - a_{i22}' + a_{i12} & a_{i33}'' - a_{i23}' + a_{i13} \end{vmatrix} = 0 \qquad (i = 1,\ 2).$$

We can now state the following conclusion:

If we have given a system of differential forms (1), *in which the coefficients* a_{ilj} *are functions of* t *of class* $C^{(l)}$ *on* (t_1t_2) *such that the matrix* $\| a_{ipj} \|$ *is of rank* n, *and if these coefficients satisfy, on that interval, the* pn *conditions* (10) $(r = 0, 1, \cdots, p-1;\ i = 1, \cdots, n)$, *then* n^2 *functions of* t, A_{ipk}, *may be chosen arbitrarily except for the condition* $| A_{ipk} | \neq 0$ *on* (t_1t_2); *and* $(p+1)n^2 + n$ *functions of* t, A_{ilk} *and* α_{kj}, *are then determined uniquely, in such a way that the forms* (1) *are reducible to the system* (2).

In the application of the reduction discussed in this note, it is frequently desirable, after its possibility has been established, to be able to select arbitrarily the coefficients α_{kj} in equations (3).

To determine the conditions under which this is possible, we substitute (3) in (2) and compare the coefficients of $x_j{}^{(p)}$ with the corresponding coefficients in (1). This leads to i systems of equations, each system consisting of $n+1$ equations in the n unknowns A_{ipk}:

$$\sum_{k=1}^{n} A_{ipk}\alpha_{kj} = a_{ipj} \qquad \begin{matrix} (i = 1, \cdots, n), \\ (j = 1, \cdots, n+1). \end{matrix}$$

These will be solvable for A_{ipk} if the i eliminants vanish. It is furthermore clear, that if α_{kj} are finite, the values of A_{ipk} so determined satisfy the condition $| A_{ipk} | \neq 0$, required in our theorem.

Thence, if conditions (10) are satisfied, the coefficients α_{kj} may be selected arbitrarily and all the coefficients of the new forms determined uniquely, but for a constant factor, if and only if we also have:

$$\sum_{j=1}^{n=1} a_{ipj} D_j = 0 \qquad (i = 1, \cdots, n), \tag{11}$$

where D_j is the determinant obtained from the matrix $| \alpha_{kj} |$ by striking out the jth column and multiplying by $(-1)^j$.

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